Journal of Approximation Theory **103**, 302–328 (2000)
doi:10.1006/jath.1999.3428, available online at http://www.idealibrary.com on IDEAL®

Entropy of $C(K)$-Valued Operators

Ingo Steinwart

Friedrich-Schiller-Universität, Ernst-Abbe-Platz 1–4, 07743 Jena, Germany
E-mail: steinwart@minet.uni-jena.de

Communicated by Allan Pinkus

Received March 19, 1999; accepted in revised form September 16, 1999

We investigate how the entropy numbers $(e_n(T))$ of an arbitrary Hölder-continuous operator $T: E \to C(K)$ are influenced by the entropy numbers $(\varepsilon_n(K))$ of the underlying compact metric space K and the geometry of E. We derive diverse universal inequalities relating finitely many $\varepsilon_n(K)$'s with finitely many $e_n(T)$'s which yield statements about the asymptotically optimal behaviour of the sequence $(e_n(T))$ in terms of the sequence $(\varepsilon_n(K))$. As an application we present new methods for estimating the entropy numbers of a precompact and convex subset in a Banach space E, provided that the entropy numbers of its extremal points are known.

1. INTRODUCTION

Let (K, d) be a metric space and $B(x, \varepsilon) := \{ y \in K : d(x, y) \leqslant \varepsilon \}$ the closed ball with radius ε and centre x. Then for a bounded subset $M \subset K$ the nth entropy number of M is defined by

$$\varepsilon_n(M) := \inf \left\{ \varepsilon > 0 : \exists x_1, ..., x_q \in K, q \leqslant n \text{ such that } M \subset \bigcup_{k=1}^{q} B(x_k, \varepsilon) \right\}$$

and the nth dyadic entropy number of M is $e_n(M) := \varepsilon_{2^{n-1}}(M)$. Furthermore, given a (bounded, linear) operator $T: E \to F$ between Banach spaces E and F the nth dyadic entropy number of T is defined by

$$e_n(T) := e_n(T(B_E)),$$

where B_E is the closed unit ball of E. For a subset A of a Banach space E we denote the absolutely convex hull of A by co$A$.

In this paper we prove several inequalities which describe how the entropy numbers of an arbitrary 1-Hölder-continuous operator $T: E \to C(K)$ (a definition can be found in the next section) are influenced by the entropy numbers of the underlying compact metric space K and the

0021-9045/00 $35.00

geometry of E. These inequalities complement earlier proven results of Carl *et al.* [5].

As an application we prove the following inequalities which hold for every Banach space E of type p, $1 < p \leqslant 2$, $\beta := 1 - 1/p$ and all precompact subsets $A \subset B_E$.

(i) For $0 < \alpha < \infty$ and $0 \leqslant \gamma < \infty$ we have

$$\sup_{k \leqslant n} k^{\alpha+\beta} (\log_2(k+1))^{\gamma}\, e_k(\mathrm{co} A) \leqslant \frac{c}{\varepsilon_1(A)} \sup_{k \leqslant n^{1+\beta/\alpha}} k^{\alpha} (\log_2(k+1))^{\gamma}\, \varepsilon_k(A).$$

(ii) For $0 < \alpha < \beta$ we have

$$\sup_{k \leqslant n} k^{\alpha} e_k(\mathrm{co} A) \leqslant \frac{c}{\varepsilon_1(A)} \sup_{k \leqslant n} k^{\alpha} e_k(A).$$

(iii) Let $\beta < \alpha < \infty$, $a_n := n^{\beta/(\alpha-\sigma)} \log_2(n+1)$ and $f\colon [0, \infty) \to (0, \infty)$ be a function with $a^{-\sigma} f(x) \leqslant f(ax) \leqslant a^{\sigma} f(x)$ for some $0 \leqslant \sigma < \alpha - \beta$ and all $a, x \geqslant 1$. Then we have

$$\sup_{k \leqslant n} k^{\beta} (\log_2(k+1))^{\alpha-\beta} f(\log_2(k+1))\, e_k(\mathrm{co} A) \leqslant \frac{c}{\varepsilon_1(A)} \sup_{k \leqslant a_n} k^{\alpha} f(k)\, e_k(A).$$

These inequalities complement results of Carl *et al.* [6], where the asymptotic behaviour of $(e_n(\mathrm{co} A))$ was considered in the case of $\varepsilon_n(A) \leqslant n^{-\alpha}$, $0 < \alpha < \infty$ and $\varepsilon_n(A) \leqslant (\log(n+1))^{-\alpha}$, $\alpha \neq \beta$.

Finally we show with the help of some particular sets that our inequalities yield asymptotically optimal results for both 1-Hölder-continuous operators and convex hulls.

2. PRELIMINARIES

For an operator $T\colon E \to \ell_\infty(A)$ the modulus of continuity $\omega(T, .)$ is defined by

$$\omega(T, \delta) := \sup_{x \in B_E} \sup_{d(s,t) \leqslant \delta} |Tx(s) - Tx(t)| \qquad (\delta > 0),$$

where (A, d) is a precompact metric space and $\ell_\infty(A)$ denotes the Banach space of all bounded number families $(\xi_t)_{t \in A}$ over A with norm

$$\|(\xi_t)\|_\infty := \sup_{t \in A} |\xi_t|.$$

An operator $T: E \to \ell_\infty(A)$ is called α-Hölder-continuous, $0 < \alpha \leqslant 1$, if

$$|T|_\alpha := \sup_{\delta > 0} \frac{\omega(T, \delta)}{\delta^\alpha} < \infty.$$

In this case we write $\|T\|\,\alpha := \max\{\|T\|, |T|_\alpha\}$. We recall that by a well-known inequality of Carl (cf. [2]) we have

$$\sup_{k \leqslant n} k^\alpha e_k(T) \leqslant c_\alpha \sup_{k \leqslant n} k^\alpha a_k(T) \tag{1}$$

for every operator $T: E \to F$ and all $\alpha > 0$, where $c_\alpha \geqslant 1$ is a constant only depending on α, and $a_k(T)$ denotes the kth approximation number of T, defined by

$$a_k(T) := \inf\{\|T - A\| : A : E \to F \text{ bounded, linear with rank } A < k\}.$$

Furthermore, for an operator $T: E \to C(K)$ we always have

$$a_{k+1}(T) \leqslant \omega(T, \varepsilon_k(K))$$

(cf. [8]). Hence, for 1-Hölder-continuous operators $T: E \to C(K)$ inequality (1) results in

$$\sup_{k \leqslant n} k^\alpha e_k(T) \leqslant 2^\alpha c_\alpha \|T\|_1 \max\{1, \sup_{k \leqslant n-1} k^\alpha \varepsilon_k(K)\}. \tag{2}$$

In the case of general Banach spaces E, this estimate is asymptotically optimal. However, if one also consider the geometry of the space E in terms of so-called "local estimates" of entropy numbers, better estimates for $(e_n(T))$ are known in the case of polynomial degree of $(\varepsilon_n(K))$ due to Carl *et al.* (cf. [5]). In this paper we prove inequalities similar to (2), which cover the results of [5], but also give asymptotically optimal estimates for various other cases, in which the order of decay of $(\varepsilon_n(K))$ is not faster than power type.

Several applications for their results are given by Carl and Edmunds in [4]. We restrict ourselves to consider the entropy numbers of the absolutely convex hull co$A$ of a given precompact subset $A \subset E$ of a B-convex Banach space E under the assumption of "known" entropy numbers of A as described in the Introduction. For this aim let $\ell_1(A)$ denote the Banach space of all summable families of real numbers $(\xi_t)_{t \in A}$ over an index set A with norm

$$\|(\xi_t)\|_1 := \sum_{t \in A} |\xi_t|.$$

Furthermore, for a precompact subset $A \subset E$ of a Banach space E let T_A be the operator $\ell_1(A) \to E$ defined by $T_A(e_t) := t$, where $(e_t)_{t \in A}$ is the canonical basis of $\ell_1(A)$. Since $\overline{\mathrm{co} A} = \overline{T(B_{\ell_1(A)})}$ we have

$$e_n(T_A) = e_n(\mathrm{co} A).$$

Moreover, T'_A as an operator mapping E' into $\ell_\infty(A)$ is 1-Hölder-continuous with $\|T'_A\|_1 = \max\{\|A\|, 1\}$, where $\|A\| := \sup_{x \in A} \|x\|$. With results of [1] this will allow us to estimate $e_n(\mathrm{co} A)$ by the first a_n entropy numbers $\varepsilon_1(A), ..., \varepsilon_{a_n}(A)$ as described above.

A Banach space E is of type p, $1 \leqslant p \leqslant 2$, if there exists a constant $c > 0$, such that for all $n \in \mathbb{N}$ and all $x_1, ..., x_n \in E$ we have

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t)\, x_i \right\|_E^2 dt \right)^{1/2} \leqslant c \left(\sum_{i=1}^n \|x_i\|_E^p \right)^{1/p},$$

where r_n is the nth Rademacher function, that is, $r_n(t) = \mathrm{sign}(\sin(2^n \pi t))$. The $\mathscr{L}_q$-spaces of Lindenstrauss and Pełczynski are of type $p = \min\{2, q\}$ for $1 \leqslant q < \infty$, in particular the Lebesgue space $L_q(\mu)$ is of type $p = \min\{2, q\}$ for $1 \leqslant q < \infty$. A Banach space is called B-convex, if it is of some type $p > 1$.

The geometry of a Banach space E will be expressed in terms of so-called "local estimates" for the entropy numbers of operators $T: E \to \ell_\infty^n$, where ℓ_∞^n denotes the n-dimensional ℓ_∞-space. Namely, we consider Banach spaces E for which for some $\beta > 0$ there exists a constant $c_\beta(E) \geqslant 1$, such that for all $n \in \mathbb{N}$ and every $T: E \to \ell_\infty^n$ we have

$$e_k(T) \leqslant c_\beta(E)\, \|T\| \left(\frac{\log_2(n/k + 1)}{k} \right)^\beta, \qquad 1 \leqslant k \leqslant n. \tag{3}$$

It was shown in [6] that this estimate is true for $\beta = 1 - \frac{1}{p}$, if E' is of type p, $1 < p \leqslant 2$, or if E' is at least of weak type p for $1 < p < 2$. The proof in [6] is based on the earlier studied "dual" situation, which states inequalities of the above type for operators $T: \ell_1^n \to E$. This was first considered by Maurey (cf. [13]), whose results were improved by Carl in [3]. Finally, Junge and M. Defant showed [9] that for $1 < p < 2$ the "dual" situation holds for $\beta = 1 - 1/p$ if and only if E is of weak type p. As mentioned in [6], the parameter β for inequalities of type (3) is bounded by $\frac{1}{2}$ since Dvoretzky's Theorem. Moreover, in the case that E is a Hilbert space and $\beta = \frac{1}{2}$, Carl and Pajor proved an analogue estimate for the Kolmogorov numbers

$$d_n(T) := \inf\{\|Q_{F_o}^F T\| : \dim F_o < n\},$$

where $Q^F_{F_o}$ denotes the canonical surjection from the Banach space F onto the quotient space F/F_o (cf. [7]). This will be used in Section 6 to estimate Gelfand widths of convex sets in Hilbert spaces.

For $0<p<\infty$ and $T\colon E\to F$ being an arbitrary operator we define

$$\lambda_{p,\infty}(T):=\sup_{n\geqslant 1} n^{1/p}e_n(T).$$

If $T_1, ..., T_n$ are operators acting between E and F with $\lambda_{p,\infty}(T_i)<\infty$, we have

$$\lambda_{p,\infty}\left(\sum_{i=1}^{n} T_i\right)\leqslant c_p\left(\sum_{i=1}^{n} (\lambda_{p,\infty}(T_i))^s\right)^{1/s},$$

where $c_p>0$ is a constant only depending on p and $s=\frac{p}{1+p}$. In the sequel we also need the following well-known fact:

Lemma 1. *Assume that the condition* (3) *is true for the Banach space E and the parameter $\beta\in(0,1/2]$. Then we have*

$$\lambda_{1/\beta,\infty}(T)\leqslant 2c_\beta(E)\,\|T\|\,(\log_2(n+1))^\beta$$

for all $T\colon E\to\ell^n_\infty$ and all $n\in\mathbb{N}$. Moreover, for every $\sigma>\beta$ there exists a constant $c_{\beta,\sigma}(E)\geqslant 1$, such that for all $n\in\mathbb{N}$ and all operators $T\colon E\to\ell^n_\infty$ the estimate

$$\lambda_{1/\sigma,\infty}(T)\leqslant c_{\beta,\sigma}(E)\,\|T\|\,n^{\sigma-\beta}$$

holds.

Finally, we need the following theorem of [1], which is an answer to the so-called duality problem of entropy numbers:

Proposition 1. *Let E be a Banach space and F be a B-convex Banach space. Then for every $\alpha>0$ there exists a constant $d_\alpha(F)\geqslant 1$, such that for all compact operators $T\colon E\to F$ and all $n\geqslant 1$ we have*

$$\frac{1}{d_\alpha(F)}\sup_{k\leqslant n} k^\alpha e_k(T)\leqslant\sup_{k\leqslant n} k^\alpha e_k(T')\leqslant d_\alpha(F)\sup_{k\leqslant n} k^\alpha e_k(T).$$

Let (a_n), (b_n) be two positive sequences. We write $a_n\preccurlyeq b_n$ if there exists a constant $c>0$ such that $a_n\leqslant c\,b_n$ for all $n\geqslant 1$. Moreover we write $a_n\sim b_n$ if $a_n\preccurlyeq b_n$ and $b_n\preccurlyeq a_n$.

As a consequence of Proposition 1 we get by a trick of Carl in [3]:

PROPOSITION 2. *Let E be a Banach space, F be a B-convex Banach space and $T: E \to F$ be a compact operator. Moreover let (a_n) be a sequence such that $(n^\alpha a_n)$ is increasing for some $\alpha > 0$. Then we have*

$$e_n(T) \preccurlyeq a_n \qquad \textit{iff} \quad e_n(T') \preccurlyeq a_n$$

and

$$e_n(T) \sim a_n \qquad \textit{iff} \quad e_n(T') \sim a_n.$$

Proof. The first assertion is a direct consequence of Proposition 1. Now we assume $e_n(T) \sim a_n$. Then we already know $e_n(T') \leqslant \rho_1 \, a_n$. Since $(n^\alpha a_n)$ is increasing we get $a_n \leqslant c^\alpha \, a_{c \cdot n}$ for all $c, n \in \mathbb{N}$. Moreover, for $\sigma > \alpha$ and suitable constants $\rho_2, \rho_3 \geqslant 1$ we obtain

$$\begin{aligned}
(c \cdot n)^\sigma \, a_{c \cdot n} &\leqslant \rho_2 (c \cdot n)^\sigma \, e_{c \cdot n}(T) \\
&\leqslant \rho_3 \sup_{k \leqslant c \cdot n} k^\sigma e_k(T') \\
&\leqslant \rho_3 (\sup_{k \leqslant n} k^\sigma e_k(T') + \sup_{n \leqslant k \leqslant c \cdot n} k^\sigma e_k(T')) \\
&\leqslant \rho_1 \rho_3 n^\sigma a_n + \rho_3 e_n(T')(c \cdot n)^\sigma
\end{aligned}$$

by Proposition 1. Hence, if we choose $c \in \mathbb{N}$ with $c^{\sigma - \alpha} > \rho_1 \; \rho_3$, we get $e_n(T') \sim a_n$. The converse implication can be proven analogously. ▮

3. THE MAIN RESULTS

In this section we state our results concerning the entropy behaviour of a given 1-Hölder-continuous operator $T: E \to C(K)$. Moreover, we give some examples, which show that the results let us obtain asymptotically optimal estimates of $(e_n(T))$.

We restrict ourselves to 1-Hölder-continuous operators, since it is easy to derive similar results for α-Hölder-continuous operators by equipping (K, d) with the new metric d^α (cf. [4]).

Moreover, all results of this section can be applied to 1-Hölder-continuous operators $T: E \to \ell_\infty(A)$, where A is a precompact metric space, since such operators factor canonically through $C(\tilde{A})$, where $\tilde{A}$ denotes the completion of A.

For brevity's sake we write $c_K := \max\{1, \varepsilon_1(K)^{-1}\}$, whenever K is a compact metric space. The major aim of this paper are the following theorems:

Theorem 1. *Let E be a Banach space such that the condition* (3) *holds for the parameter $\beta \in (0, 1/2]$. Then for all $0 < \alpha < \beta$ there exists a constant $c_{\alpha,\beta}(E) \geqslant 1$, such that for all compact metric spaces (K, d), all 1-Hölder-continuous operators $T: E \to C(K)$ and all $n \in \mathbb{N}$ we have*

$$\sup_{k \leqslant n} k^{\alpha} e_k(T) \leqslant c_{\alpha,\beta}(E)\, c_K \, \|T\|_1 \sup_{k \leqslant n} k^{\alpha} e_k(K).$$

Theorem 2. *Let E be a Banach space such that the condition* (3) *holds for the parameter $\beta \in (0, 1/2]$. Furthermore let $0 \leqslant \sigma < \alpha - \beta$ and $f: [0, \infty) \to (0, \infty)$ be a function such that*

$$a^{-\sigma} f(s) \leqslant f(a \cdot s) \leqslant a^{\sigma} f(s) \tag{4}$$

holds for all $a, s \geqslant 1$. Then there exists a constant $c \geqslant 1$, such that for all compact metric spaces (K, d), all 1-Hölder-continuous operators $T: E \to C(K)$ and all $n \in \mathbb{N}$ the inequality

$$\sup_{k \leqslant n} k^{\beta} (\log_2(k+1))^{\alpha-\beta} f(\log_2(k+1))\, e_k(T) \leqslant c \cdot c_K \, \|T\|_1 \sup_{k \leqslant a_n} k^{\alpha} f(k)\, e_k(K)$$

holds with $a_n := n^{\beta/(\alpha-\sigma)} \log_2(n+1)$.

Theorem 3. *Let E be a Banach space such that the condition* (3) *holds for the parameter $\beta \in (0, 1/2]$. Then for all $\alpha > 0$ and $\gamma \geqslant 0$ there exists a constant $c_{\alpha,\beta,\gamma}(E) \geqslant 1$, such that for all compact metric spaces (K, d), all 1-Hölder-continuous operators $T: E \to C(K)$ and all $n \in \mathbb{N}$ we have*

$$\sup_{k \leqslant n} k^{\alpha+\beta} (\log_2(k+1))^{\gamma}\, e_k(T) \leqslant c_{\alpha,\beta,\gamma}(E)\, c_K \, \|T\|_1 \sup_{k \leqslant n^{1+\beta/\alpha}} k^{\alpha} (\log_2(k+1))^{\gamma}\, \varepsilon_k(K).$$

In particular the estimates of the above theorems hold, if E' is of type $p > 1$ and $\beta := 1 - \frac{1}{p}$.

Corollary 1. *Let E be a Banach space, such that E' is of type $p > 1$ and $\beta := 1 - \frac{1}{p}$. If (K, d) is a compact metric space such that*

$$e_n(K) \preccurlyeq n^{-\alpha} \, (\log(n+1))^{-\gamma}$$

holds for the dyadic *entropy numbers and some $\alpha \neq \beta$, $\gamma \in \mathbb{R}$, then for all 1-Hölder-continuous operators $T: E \to C(K)$ we have*

$$e_n(T) \preccurlyeq \begin{cases} n^{-\alpha} (\log(n+1))^{-\gamma} & \text{if } \; 0 < \alpha < \beta \\ n^{-\beta} (\log(n+1))^{-\alpha+\beta} \, (\log(\log(n+1)+1))^{-\gamma} & \text{if } \; \beta < \alpha < \infty. \end{cases}$$

Proof. The case $0<\alpha<\beta$ immediately follows from Theorem 1. If $\beta<\alpha<\infty$ let $c:=\exp(\frac{2}{\alpha-\beta})$ and $f(x):=(\log(c\cdot(x+1)))^{\gamma}$. Then we have

$$a^{-\sigma}f(x)\leqslant f(ax)\leqslant a^{\sigma}f(x)$$

for $a, x\geqslant 1$ and $\sigma:=\frac{\alpha-\beta}{2}$. Since $f(x)\sim(\log(x+1))^{\gamma}$ for $x\to\infty$ the assertion follows by Theorem 2. ▮

As a consequence of Theorem 3 we obtain the following corollary, which was proven for $\gamma=0$ in [5] and for $\gamma\leqslant 0$ in [11]:

COROLLARY 2. *Let E be a Banach space, such that E' is of type $p>1$ and $\beta:=1-\frac{1}{p}$. If (K, d) is a compact metric space such that*

$$\varepsilon_n(K)\preccurlyeq n^{-\alpha}(\log(n+1))^{-\gamma}$$

holds for some $\alpha>0$ and $\gamma\in\mathbb{R}$, then for all 1-Hölder-continuous operators $T\colon E\to C(K)$ we have

$$e_n(T)\preccurlyeq n^{-\alpha-\beta}(\log(n+1))^{-\gamma}.$$

Proof. The case $\gamma\geqslant 0$ follows immediately from Theorem 3. On the other hand for $\gamma<0$ we have

$$\begin{aligned} n^{\alpha+\beta}(\log_2(n+1))^{-\gamma}\,e_n(T) &\preccurlyeq \sup_{k\leqslant n^{1+\beta/\alpha}} k^{\alpha}(\log_2(k+1))^{-\gamma}\,\varepsilon_k(K) \\ &\preccurlyeq \sup_{k\leqslant n^{1+\beta/\alpha}} (\log_2(k+1))^{-\gamma}\,(\log_2(k+1))^{-\gamma} \\ &\preccurlyeq (\log_2(n+1))^{-2\gamma}. \end{aligned}$$ ▮

Remarks. We will see in the last section, that the estimates of the corollaries are asymptotically optimal for $E=\ell_p$, $2\leqslant p<\infty$.

Moreover, the theorems of this section are also valid for the Kolmogorov numbers $d_n(T\colon E\to C(K))$ if E is a Hilbert space, since the essential condition (3) is true for them in this case.

The logarithmic term $(\log(n+1))^{-\gamma}$ in the corollaries can be replaced by various other functions, e.g., $(\log(n+1))^{-\gamma}\ (\log(\log(n+1)+1))^{-\eta}$ for $\gamma, \eta\in\mathbb{R}$.

Finally we remark, that with the techniques and notations of Theorem 1 we obtain

$$\sup_{k\leqslant n} k^{\beta}(\log(k+1))^{-(1+\beta)}\,e_k(T)\leqslant c_{\beta}(E)\,c_K\,\|T\|_1 \sup_{k\leqslant n} k^{\beta}e_k(K) \tag{5}$$

for the limiting case $\alpha=\beta$.

4. DECOMPOSITION OF OPERATORS

To prove the Theorems 1, 2, and 3, we need some technical facts about the decomposition of 1-Hölder-continuous operators $T: E \to C(K)$ into finite sums of type

$$T = \sum_{i=1}^{n} T_i + S,$$

where the operators T_i have finite rank and some additional properties. Therefore we define

$$\operatorname{supp} \varphi := \{t \in K : \varphi(t) \neq 0\}$$

for $\varphi \in C(K)$. Moreover, we write $E \stackrel{1}{=} F$ or $E \stackrel{1}{\hookrightarrow} F$, if the Banach space E is isometrically isomorphic to F, resp. isometrically embedded into F.

We start with a simple lemma, whose proof we omit:

LEMMA 2. *Let (K, d) be a compact metric space and $T: E \to C(K)$ be a 1-Hölder-continuous operator. Furthermore, let $\varphi_1, ..., \varphi_n \in C(K)$ be a partition of unity and $t_1, ..., t_n \in K$ such that $\varphi_i(t_j) = \delta_{ij}$. Then for the operator*

$$A: E \to C(K)$$

$$x \mapsto \sum_{i=1}^{n} Tx(t_i)\, \varphi_i$$

the following statements hold:

$$\|A\| \leqslant \|T\|$$

$$\|T - A\| \leqslant 2 \sup_{i \leqslant n} \varepsilon_1(\mathit{supp}\ \varphi_i)\ \|T\|_1$$

$$\operatorname{range} A \subset \operatorname{span}\{\varphi_1, ..., \varphi_n\} \stackrel{1}{=} \ell_\infty^n.$$

The following lemma constructs a special partition of unity from a given one.

LEMMA 3. *Let (K, d) be a compact metric space, $(\varphi_i) \subset C(K)$ be a partition of unity and $t_i \in K$, such that $\varphi_i(t_j) = \delta_{ij}$. Furthermore let $M \in \mathbb{N}$ and $\delta > 0$ such that $\varepsilon_M(K) < \delta$. Then there exist a partition of unity (ψ_i) of at most M functions and $s_i \in K$ such that*

$$\psi_i(s_j) = \delta_{ij}$$

$$\varepsilon_1(\mathit{supp}\ \psi_i) \leqslant \delta + \sup_j \varepsilon_1(\mathit{supp}\ \varphi_j)$$

$$\operatorname{span}(\psi_i) \subset \operatorname{span}(\varphi_i).$$

Proof. Let $\varepsilon := \sup_j \varepsilon_1(\operatorname{supp} \varphi_j)$. Since K is compact, there are elements $y_i \in K$ such that $\operatorname{supp} \varphi_i \subset B(y_i, \varepsilon)$. Furthermore, $\varepsilon_M(K) < \delta$ implies the existence of a δ-net $\{z_1, ..., z_m\} \subset K$ with $m \leqslant M$. Now let $A_1, ..., A_m$ be a partition of K with $A_i \subset B(z_i, \delta)$. Then for $1 \leqslant i \leqslant m$ we define

$$\psi_i := \sum_{y_j \in A_i} \varphi_j$$

if there exists an index j with $y_j \in A_i$. Otherwise we omit the index i. Therefore, (ψ_i) is a partition of unity of at most M functions and $\operatorname{span}(\psi_i) \subset \operatorname{span}(\varphi_i)$. Moreover, for $t \in \operatorname{supp} \psi_i$ there exists $y_j \in A_i$ such that $t \in \operatorname{supp} \varphi_j \subset B(y_j, \varepsilon)$. Hence $d(t, y_j) \leqslant \varepsilon$. On the other hand, $y_j \in A_i \subset B(z_i, \delta)$ implies $d(y_j, z_i) \leqslant \delta$. Therefore, we get $d(t, z_i) \leqslant \delta + \varepsilon$ and hence

$$\varepsilon_1(\operatorname{supp} \psi_i) \leqslant \delta + \varepsilon.$$

Finally, let $1 \leqslant i \leqslant m$ such that there exists an index j with $y_j \in A_i$. Define $s_i := t_j$. Then for $k \leqslant m$ we obtain

$$\psi_k(s_i) = \sum_{y_l \in A_k} \varphi_l(s_i) = \sum_{y_l \in A_k} \varphi_l(t_j) = \sum_{y_l \in A_k} \delta_{l,j} = \delta_{i,k}$$

since $y_j \in A_k$ if and only if $i = k$. ∎

Iterating the procedure of Lemma 3 we receive:

LEMMA 4. *Let (K, d) be a compact metric space and $n \geqslant 1$ be an integer. Moreover, let $\alpha_o, \alpha_1, ..., \alpha_n \in \mathbb{N}$ and $\beta_{-1}, \beta_o, \beta_1, ..., \beta_n > 0$ be finite sequences, such that*

$$\varepsilon_{\alpha_i}(K) < \beta_i$$

$$2\beta_i \leqslant \beta_{i-1}$$

for all $0 \leqslant i \leqslant n$. Then there exist partitions of unity $P_o, ..., P_n \subset C(K)$, $P_k = (\varphi_{k,i})_i$ and elements $t_{k,i} \in K$ such that for all $0 \leqslant k \leqslant n$ the following statements are true:

$$\operatorname{card} P_k \leqslant \alpha_k$$

$$\varepsilon_1(\mathit{supp}\ \varphi_{k,i}) \leqslant \beta_{k-1}$$

$$\varphi_{k,i}(t_{k,j}) = \delta_{ij}$$

$$\operatorname{span} P_k \subset \operatorname{span} P_{k+1}.$$

Proof. Let $k=n$. Since $\varepsilon_{\alpha_n}(K)<\beta_n$ there exists a minimal β_n-net Γ consisting of $a_n \leqslant \alpha_n$ elements $x_1, ..., x_{a_n}$. Let $P_n=(\varphi_{n,i})_{i\leqslant a_n}$ be a partition of unity subordinate to this open covering. Then

$$\varepsilon_1(\operatorname{supp}\varphi_{n,i}) \leqslant \varepsilon_1(B(x_i, \beta_n)) = \beta_n \leqslant \beta_{n-1}$$

and since Γ is minimal, we can find elements $t_{n,j}\in B(x_j, \beta_n)$ such that $t_{n,j}\notin B(x_i, \beta_n)$ for $i\neq j$. Hence $\varphi_{n,i}(t_{n,j})=\delta_{ij}$.

Now we assume, that we have already constructed P_{k+1} according to the assertion. Then by Lemma 3 we get a partition of unity $P_k:=(\psi_i)$ of at most α_k functions and elements $(s_i)_i$ such that

$$\varepsilon_1(\operatorname{supp}\psi_i) \leqslant \beta_k + \sup_j \varepsilon_1(\operatorname{supp}\varphi_{k+1,j}) \leqslant 2\beta_k \leqslant \beta_{k-1}$$

$$\psi_i(s_j)=\delta_{ij}$$

$$\operatorname{span} P_k \subset \operatorname{span} P_{k+1}.$$

Therefore, we define $\varphi_{k,i}:=\psi_i$ and $t_{k,i}:=s_i$. ▮

Now we combine the previous lemma with Lemma 2 and obtain a decomposition of 1-Hölder-continuous operators $T: E\to C(K)$ generalizing a corresponding decomposition in [5]:

LEMMA 5. *Let (K, d) be a compact metric space and $n\geqslant 1$ be an integer. Moreover, let $\alpha_o, \alpha_1, ..., \alpha_n\in\mathbb{N}$, $\beta_{-1}, \beta_o, \beta_1, ..., \beta_n>0$ be finite sequences, such that*

$$\varepsilon_{\alpha_i}(K)<\beta_i$$

$$2\beta_i \leqslant \beta_{i-1}$$

for all $0\leqslant i\leqslant n$. Furthermore, let $T: E\to C(K)$ be a 1-Hölder-continuous operator. Then there exists a decomposition

$$T=\sum_{i=0}^{n} T_i + S$$

by operators $T_i: E\to C(K)$ and $S: E\to C(K)$, such that

$$\|T_i\| \leqslant 4\beta_{i-2}\,\|T\|_1 \qquad \textit{for} \quad i=1, ..., n$$

$$\|T_o\| \leqslant \|T\|$$

$$\|S\| \leqslant 2\beta_{n-1}\,\|T\|_1$$

$$\operatorname{range} T_i \overset{1}{\hookrightarrow} \ell_\infty^{\alpha_i} \qquad \textit{for} \quad i=0, ..., n.$$

Moreover, T_o is of the form $x \mapsto \sum_i Tx(s_i)\,\psi_i(.)$, where $(\psi_i)_i \subset C(K)$ is a partition of unity of at most α_o functions with $\psi_i(s_j) = \delta_{i,j}$ and $\varepsilon_1(supp\ \psi_i) \leqslant \beta_{-1}$.

Proof. By Lemma 4 we get partitions of unity $P_0, ..., P_n$. Therefore, by Lemma 2 we can construct operators $A_k: E \to C(K)$ with

$$\|A_k\| \leqslant \|T\|$$

$$\|T - A_k\| \leqslant 2\beta_{k-1}\ \|T\|_1$$

$$\text{range } A_k \subset \text{span } P_k \overset{1}{\hookrightarrow} \ell_\infty^{\alpha_k}.$$

Now we define $T_o := A_o$, $T_i := A_i - A_{i-1}$ for $i = 1, ..., n$ and $S := T - A_n$. Clearly, we have $T = \sum_{i=0}^n T_i + S$ and $\|S\| \leqslant 2\beta_{n-1}\ \|T\|$. Furthermore,

$$\|T_i\| \leqslant \|T - A_i\| + \|T - A_{i-1}\| \leqslant 2\ \|T\|_1\ (\beta_{i-1} + \beta_{i-2}) \leqslant 4\beta_{i-2}\ \|T\|_1$$

holds. Since range $A_{k-1} \subset$ span P_k, we finally obtain range $T_k \subset$ span P_k. ∎

5. THE PROOFS OF THE MAIN THEOREMS

Proof of Theorem 1. We first assume that $\varepsilon_1(K) \geqslant 1$. Then for $n = 1$ the assertion is trivial. Therefore let us additionally assume $n \geqslant 2$. For fixed $0 < \alpha < \beta$ we define

$$C := \sup_{k \leqslant n} k^\alpha e_k(K)$$

and $r := \lfloor \alpha \log_2(n-1) \rfloor$. To apply Lemma 5, we use the finite sequences

$$\alpha_i := \lfloor \exp_2 2^{i/\alpha} \rfloor \qquad \text{for} \quad 0 \leqslant i \leqslant r$$

$$\beta_i := C \cdot 2^{-i+\alpha} \qquad \text{for} \quad -2 \leqslant i \leqslant r.$$

Since $\lfloor \log_2 \alpha_i \rfloor + 1 \leqslant n$ one easily verifies $\varepsilon_{\alpha_i}(K) < \beta_i$ for $1 \leqslant i \leqslant r$. Additionally $\varepsilon_{\alpha_o}(K) = e_2(K) \leqslant C \cdot 2^{-\alpha} < \beta_o$ and $2\beta_i \leqslant \beta_{i-1}$ hold. Hence we can find a decomposition

$$T = \sum_{i=0}^r T_i + S$$

according to Lemma 5. Thus, for $s := \frac{1}{1+\beta}$ and a suitable $c_{1/\beta} \geqslant 1$ we obtain

$$\begin{aligned}
n^{\beta} e_n(T) &\leqslant n^{\beta} e_n\left(\sum_{i=0}^{r} T_i\right) + n^{\beta} \|S\| \\
&\leqslant \sup_{j \geqslant 1} j^{\beta} e_j\left(\sum_{i=0}^{r} T_i\right) + 2n^{\beta} \beta_{r-1} \|T\|_1 \\
&\leqslant c_{1/\beta}\left(\sum_{i=0}^{r} (\lambda_{1/\beta,\infty}(T_i))^s\right)^{1/s} + C2^{2+\alpha+\beta}(n-1)^{\beta}\, 2^{-r} \|T\|_1.
\end{aligned}$$

Since $4\beta_{-2} = 4C2^{2+\alpha} \geqslant 1$ we observe $\|T_o\| \leqslant \|T\| \leqslant \|T\|_1 \leqslant 4\beta_{-2} \|T\|_1$. Hence for $0 \leqslant i \leqslant r$ we may estimate

$$\begin{aligned}
\lambda_{1/\beta,\infty}(T_i) &\leqslant 2c_{\beta}(E)\, \|T_i\|\, (\log_2(\alpha_i+1))^{\beta} \\
&\leqslant 2^4 c_{\beta}(E)\, \beta_{i-2}\, \|T\|_1\, (\log_2 \alpha_i)^{\beta} \\
&= 2^{6+\alpha} c_{\beta}(E)\, C\, \|T\|_1\, 2^{i(\beta/\alpha - 1)}
\end{aligned}$$

by Lemma 1. Thus with $c^{(1)}_{\alpha,\beta}(E) := 2^{6+\alpha} c_{1/\beta} c_{\beta}(E)$ and $c_{\alpha,\beta} := 2^{\beta/\alpha - 1}/(2^{s(\beta/\alpha-1)} - 1)^{1/s}$ we receive

$$\begin{aligned}
c_{1/\beta}\left(\sum_{i=0}^{r} (\sup_{j\geqslant 1} j^{\beta} e_j(T_i))^s\right)^{1/s} &\leqslant c^{(1)}_{\alpha,\beta}(E)\, C\, \|T\|_1 \left(\sum_{i=0}^{r} (2^{i(\beta/\alpha-1)})^s\right)^{1/s} \\
&\leqslant c^{(1)}_{\alpha,\beta}(E)\, c_{\alpha,\beta} C\, \|T\|_1\, 2^{r(\beta/\alpha-1)}.
\end{aligned}$$

Hence for $c^{(2)}_{\alpha,\beta}(E) := c^{(1)}_{\alpha,\beta}(E)\, c_{\alpha,\beta} + 2^{3+\alpha+\beta}$ we obtain

$$\begin{aligned}
n^{\beta} e_n(T) &\leqslant c^{(1)}_{\alpha,\beta}(E)\, c_{\alpha,\beta} C\, \|T\|_1\, 2^{r(\beta/\alpha-1)} + C2^{2+\alpha+\beta}(n-1)^{\beta}\, 2^{-r} \|T\|_1 \\
&\leqslant \|T\|_1\, C(c^{(1)}_{\alpha,\beta}(E)\, c_{\alpha,\beta}\, 2^{\alpha \log_2(n-1)(\beta/\alpha-1)} \\
&\quad + 2^{2+\alpha+\beta}(n-1)^{\beta}\, 2^{-\alpha \log_2(n-1)+1}) \\
&= c^{(2)}_{\alpha,\beta}(E)\, \|T\|_1\, C(n-1)^{\beta-\alpha},
\end{aligned}$$

i.e., the assertion is proven in the case $\varepsilon_1(K) \geqslant 1$ with the constant $c_{\alpha,\beta}(E) := c^{(2)}_{\alpha,\beta}(E)$. Now let us assume that $\varepsilon_1(K) < 1$. Then $\tilde{d}(s,t) := \varepsilon_1(K)^{-1}\, d(s,t)$ defines a new, equivalent metric on K with $\varepsilon_n^{(\tilde{d})}(K) = \varepsilon_1(K)^{-1}\, \varepsilon_n^{(d)}(K)$ and

$$|T|_1^{(\tilde{d})} = \varepsilon_1(K)\, |T|_1^{(d)} \leqslant |T|_1^{(d)}.$$

Hence we obtain $\varepsilon_1^{(\tilde{d})}(K)=1$ and $\|T\|_1^{(\tilde{d})} \leqslant \|T\|_1^{(d)}$. Finally we receive

$$n^\alpha e_n(T) \leqslant c_{\alpha,\beta}(E)\, \|T\|_1^{(\tilde{d})} \sup_{k \leqslant n} k^\alpha e_k^{(\tilde{d})}(K)$$

$$\leqslant c_{\alpha,\beta}(E)\, \varepsilon_1^{(d)}(K)^{-1}\, \|T\|_1^{(d)} \sup_{k \leqslant n} k^\alpha e_k^{(d)}(K). \quad \blacksquare$$

For the proof of Theorem 2 we need an additional lemma:

LEMMA 6. *Let E be a Banach space, such that the condition* (3) *holds for the parameter $\beta \in (0, 1/2]$, and let $f\colon [0, \infty) \to (0, \infty)$ be a function, such that* (4) *holds for some $0 \leqslant \sigma < \alpha - \beta$. Furthermore, let K be a compact metric space with $\varepsilon_1(K) \geqslant 1$, $T\colon E \to C(K)$ be a* 1*-Hölder-continuous operator and $\varphi_1, ..., \varphi_m \in C(K)$ be a partition of unity with $\varphi_i(t_j) = \delta_{i,j}$ for suitable elements $t_j \in K$. Then for every integer $n \geqslant 2$ and every operator*

$$A\colon E \to C(K)$$

$$x \mapsto \sum_{i=1}^{m} Tx(t_i)\, \varphi_i$$

with $m \leqslant n$ we have

$$e_n(A) \leqslant c\, \|T\|_1\, n^{-\beta}$$

$$\times \left(\sup_{i \leqslant m} \varepsilon_1(supp\ \varphi_i) + \frac{\sup_{i \leqslant n} (\log_2 (i+1))^\alpha f(\log_2 i)\, \varepsilon_i(K)}{(\log_2 n)^\alpha f(\log_2 n)} \right),$$

where $c =: c_{\alpha,\beta,f}(E)$ is a constant only depending on α, β, f and E.

Proof of Lemma 6. Let $c_1 \in \mathbb{N}$ with $c_1 > 6$ and $2^{4+\alpha+\sigma} c_1^\beta\, 2^{-c_1/6} \leqslant 1/2$. We define

$$C_n := \sup_{i \leqslant n} (\log_2(i+1))^\alpha f(\log_2 i)\, \varepsilon_i(K) \qquad \text{for} \quad n \geqslant 2$$

$$c_2 := c_1^2$$

$$c_3 := c_2^\beta \max\{1, f(1) \cdot (f(0))^{-1} \cdot (\log_2 c_2)^{\alpha+\sigma}\}$$

$$c_{\alpha,\beta,f}(E) := \max\{c_3, 3^\beta 2^{4+\alpha+\beta+\sigma} c_\beta(E)\}.$$

We proceed by induction on n. For $2 \leqslant n \leqslant c_2$ and A according to the assertion we have

$$
\begin{aligned}
1 &\leqslant (\log_2 c_2)^\alpha \max_{2 \leqslant i \leqslant c_2} f(\log_2 i)(\log_2 n)^{-\alpha} (f(\log_2 n))^{-1} \\
&\leqslant (\log_2 c_2)^\alpha \max_{2 \leqslant i \leqslant c_2} (\log_2 i)^\sigma f(1)(\log_2 n)^{-\alpha} (f(\log_2 n))^{-1} \\
&\leqslant \sup_{i \leqslant m} \varepsilon_1(\operatorname{supp} \varphi_i) + (\log_2 c_2)^{\alpha+\sigma} f(1)(\log_2 n)^{-\alpha} (f(\log_2 n))^{-1} (f(0))^{-1} C_n \\
&\leqslant \frac{c_3}{c_2^\beta} (\sup_{i \leqslant m} \varepsilon_1(\operatorname{supp} \varphi_i) + (\log_2 n)^{-\alpha} (f(\log_2 n))^{-1} C_n),
\end{aligned}
$$

since $C_n \geqslant f(0)$. Therefore we receive

$$
e_n(A) \leqslant c_{\alpha,\beta,f}(E) \, \|T\|_1 \, n^{-\beta} \left(\sup_{i \leqslant m} \varepsilon_1(\operatorname{supp} \varphi_i) + \frac{C_n}{(\log_2 n)^\alpha f(\log_2 n)} \right).
$$

Now let $n > c_2$ and A be according to the assertion. We define $M := \lfloor n/c_1 \rfloor + 1$ and $\varepsilon := \sup_{i \leqslant m} \varepsilon_1(\operatorname{supp} \varphi_i)$. Since $c_1 < \sqrt{n}$, we have $\frac{1}{2} \log_2 n \leqslant \log_2 M < \log_2 n$. Hence we obtain

$$
\begin{aligned}
(\log_2 M)^{-\alpha} (f(\log_2 M))^{-1} &\leqslant 2^\alpha (\log_2 n)^{-\alpha} \left(\frac{\log_2 n}{\log_2 M} \right)^\sigma (f(\log_2 n))^{-1} \\
&\leqslant 2^{\alpha+\sigma} (\log_2 n)^{-\alpha} (f(\log_2 n))^{-1}. \qquad (6)
\end{aligned}
$$

We set $\delta := C_n (\log_2 M)^{-\alpha} (f(\log_2 M))^{-1}$. Since $M < n$ we get

$$
\varepsilon_M(K) \leqslant C_n (\log_2(M+1))^{-\alpha} (f(\log_2 M))^{-1} < \delta.
$$

Thus by Lemma 3 there exists a partition of unity $(\psi_i) \subset C(K)$ of $k \leqslant M$ functions and elements $s_i \in K$ such that

$$
\begin{aligned}
\psi_i(s_j) &= \delta_{ij} \\
\varepsilon_1(\operatorname{supp} \psi_i) &\leqslant \delta + \varepsilon \\
\operatorname{span}(\psi_i) &\subset \operatorname{span}(\varphi_i).
\end{aligned}
$$

Now we define the operators

$$
\begin{aligned}
B \colon E &\to C(K) \\
x &\mapsto \sum_{i=1}^k Tx(s_i) \, \psi_i
\end{aligned}
$$

and $S := A - B$. Then for $r := \lfloor n/2 \rfloor$ we get

$$e_n(A) \leqslant e_r(B) + e_r(S).$$

First we estimate $e_r(B)$. Since $M < n$ we may conclude

$$\begin{aligned} e_M(B) &\leqslant c_{\alpha,\beta,f}(E)\ \|T\|_1\ M^{-\beta}\left((\delta + \varepsilon) + \frac{C_M}{(\log_2 M)^\alpha f(\log_2 M)}\right) \\ &\leqslant c_{\alpha,\beta,f}(E)\ \|T\|_1 \left(\frac{n}{c_1}\right)^{-\beta}\left(\varepsilon + \frac{2^{1+\alpha+\sigma} C_n}{(\log_2 n)^\alpha f(\log_2 n)}\right) \\ &\leqslant c_{\alpha,\beta,f}(E)\ 2^{1+\alpha+\sigma} c_1^\beta\ \|T\|_1\ n^{-\beta}\left(\varepsilon + \frac{C_n}{(\log_2 n)^\alpha f(\log_2 n)}\right) \end{aligned}$$

by the induction hypothesis and inequality (6). Furthermore, $6 < c_1 < n$ implies

$$\frac{r}{M} = \frac{\lfloor n/2 \rfloor}{\lfloor n/c_1 \rfloor + 1} \geqslant \frac{n/2 - 1}{n/c_1 + 1} = c_1 \cdot \frac{1/2 - 1/n}{1 + c_1/n} \geqslant c_1 \cdot \frac{1/2 - 1/6}{2} = \frac{c_1}{6}.$$

Thus with [8, Lemma 5.10.3] we obtain

$$\begin{aligned} e_r(B) &\leqslant 8 \cdot 2^{-r/M} e_M(B) \\ &\leqslant 8 \cdot 2^{-c_1/6} c_{\alpha,\beta,f}(E)\ 2^{1+\alpha+\sigma} c_1^\beta\ \|T\|_1\ n^{-\beta}\left(\varepsilon + \frac{C_n}{(\log_2 n)^\alpha f(\log_2 n)}\right) \\ &\leqslant \frac{1}{2}\, c_{\alpha,\beta,f}(E)\ \|T\|_1\ n^{-\beta}\left(\varepsilon + \frac{C_n}{(\log_2 n)^\alpha f(\log_2 n)}\right). \end{aligned}$$

To estimate $e_r(S)$ we first observe that

$$\begin{aligned} \|S\| &\leqslant \|T - A\| + \|T - B\| \\ &\leqslant 2\varepsilon\ \|T\|_1 + 2(\varepsilon + \delta)\ \|T\|_1 \\ &\leqslant 4\ \|T\|_1 \left(\varepsilon + \frac{C_n}{(\log_2 M)^\alpha f(\log_2 M)}\right) \\ &\leqslant 2^{2+\alpha+\sigma}\ \|T\|_1 \left(\varepsilon + \frac{C_n}{(\log_2 n)^\alpha f(\log_2 n)}\right) \end{aligned}$$

by Lemma 2. Since range S is isometrically embedded in $\ell_\infty^m \overset{1}{\hookrightarrow} \ell_\infty^n$, we finally obtain

$$\begin{aligned} e_r(S) &\leqslant 2c_\beta(E)\left(\frac{\log(n/r+1)}{r}\right)^\beta \|S\| \\ &\leqslant 2^{3+\alpha+\sigma} c_\beta(E)\left(\frac{\log_2 4}{n/3}\right)^\beta \|T\|_1 \left(\varepsilon + \frac{C_n}{(\log_2 n)^\alpha f(\log_2 n)}\right) \\ &\leqslant \frac{1}{2} c_{\alpha,\beta,f}(E)\, \|T\|_1\, n^{-\beta}\left(\varepsilon + \frac{C_n}{(\log_2 n)^\alpha f(\log_2 n)}\right) \end{aligned}$$

by condition (3). ∎

Proof of Theorem 2. As in the proof of Theorem 1 it suffices to consider the case $\varepsilon_1(K) \geqslant 1$. For fixed α, β and f we set $\gamma := \frac{\beta}{\alpha-\sigma}$ and choose an integer q with $q > 2^{1/(\alpha-\sigma)}$. Additionally, for a fixed $n \in \mathbb{N}$ with $n \geqslant a := \max\{2^q, q^{3/\gamma}\}$ we define

$$\begin{aligned} C &:= \sup_{k \leqslant n^\gamma \log_2(n+1)} k^\alpha f(k)\, e_k(K) \\ r &:= \lfloor \gamma \log_q n \rfloor - 1 \\ \alpha_i &:= n^{q^i} \qquad \text{for} \quad i = 0, ..., r \\ \beta_i &:= 2^\sigma C (\log_2 \alpha_i)^{-\alpha} (f(\log_2 \alpha_i))^{-1} \qquad \text{for} \quad i = -1, ..., r. \end{aligned}$$

An easy computation shows $\lfloor \log_2 \alpha_i \rfloor + 1 \leqslant n^\gamma \log_2(n+1)$ and hence we get

$$\begin{aligned} \varepsilon_{\alpha_i}(K) &\leqslant C(\lfloor \log_2 \alpha_i \rfloor + 1)^{-\alpha} (f(\lfloor \log_2 \alpha_i \rfloor + 1))^{-1} \\ &< C(\log_2 \alpha_i)^{-\alpha} \left(\frac{\lfloor \log_2 \alpha_i \rfloor + 1}{\log_2 \alpha_i}\right)^\sigma (f(\log_2 \alpha_i))^{-1} \\ &\leqslant 2^\sigma C (\log_2 \alpha_i)^{-\alpha} (f(\log_2 \alpha_i))^{-1} = \beta_i \end{aligned}$$

for $0 \leqslant i \leqslant r$. Furthermore, by the definition of q we obtain

$$\frac{2\beta_i}{\beta_{i-1}} = 2q^{-\alpha} \frac{f(q^{i-1} \log_2 n)}{f(q^i \log_2 n)} \leqslant 2q^{-(\alpha-\sigma)} \leqslant 1.$$

Therefore, by Lemma 5 we can decompose T in

$$T = T_o + \sum_{i=1}^r T_i + S$$

and receive

$$e_{2n}(T) \leqslant e_n(T_o) + e_n\left(\sum_{i=1}^{r} T_i\right) + e_1(S). \tag{7}$$

First we estimate the term $e_n(T_o)$. By Lemma 5 the operator T_o is constructed by a partition of unity $(\psi_i) \subset C(K)$ of at most $\alpha_o = n$ functions with

$$\begin{aligned}\varepsilon_1(\operatorname{supp} \psi_i) &\leqslant \beta_{-1} = 2^\sigma C q^\alpha (\log_2 n)^{-\alpha} (f(q^{-1} \log_2 n))^{-1} \\ &\leqslant c_{\alpha, \beta, f} C (\log_2 n)^{-\alpha} (f(\log_2 n))^{-1},\end{aligned}$$

where $c_{\alpha, \beta, f} := 2^\sigma q^{\alpha+\sigma}$. Hence by Lemma 6 there exists a constant $c^{(1)}_{\alpha, \beta, f}(E)$ such that

$$\begin{aligned}e_n(T_o) &\leqslant c^{(1)}_{\alpha, \beta, f}(E)\, \|T\|_1\, C n^{-\beta} (\log_2 n)^{-\alpha} (f(\log_2 n))^{-1} \\ &\leqslant c^{(1)}_{\alpha, \beta, f}(E)\, \|T\|_1\, C n^{-\beta} (\log_2 n)^{-\alpha+\beta} (f(\log_2 n))^{-1}.\end{aligned}$$

Now we discuss $e_1(S)$. By Lemma 5 we know that

$$\begin{aligned}e_1(S) &\leqslant 2\beta_{r-1}\, \|T\|_1 \\ &= 2^{1+\sigma} C\, \|T\|_1\, q^{-\alpha(r-1)} (\log_2 n)^{-\alpha} (f(q^{r-1} \log_2 n))^{-1} \\ &\leqslant 2^{1+\sigma} C\, \|T\|_1\, q^{-\alpha(r-1)} (\log_2 n)^{-\alpha}\, q^{\sigma(r-1)} (f(\log_2 n))^{-1} \\ &\leqslant c^{(2)}_{\alpha, \beta, f}(E)\, C\, \|T\|_1\, n^{-\beta} (\log_2 n)^{-\alpha+\beta} (f(\log_2 n))^{-1},\end{aligned}$$

where $c^{(2)}_{\alpha, \beta, f}(E) := 2^{1+\sigma}\, q^{3(\alpha-\sigma)}$. Finally we estimate $e_n(\sum_{i=1}^r T_i)$. For $s := \frac{1}{1+\beta}$ and suitable $c_{1/\beta} \geqslant 1$ we obtain

$$\begin{aligned}n^\beta e_n\left(\sum_{i=1}^{r} T_i\right) &\leqslant c_{1/\beta} \left(\sum_{i=1}^{r} (\lambda_{1/\beta, \infty}(T_i))^s\right)^{1/s} \\ &\leqslant 8 c_{1/\beta} c_\beta(E)\, \|T\|_1 \left(\sum_{i=1}^{r} ((\log_2 \alpha_i)^\beta \beta_{i-2})^s\right)^{1/s} \\ &= 2^{3+\sigma} c_{1/\beta} c_\beta(E)\, q^{2\alpha}\, \|T\|_1\, C \\ &\quad \times \left(\sum_{i=1}^{r} (q^{-i(\alpha-\beta)} (\log_2 n)^{\beta-\alpha} (f(q^{i-2} \log_2 n))^{-1})^s\right)^{1/s} \\ &\leqslant 2^{3+\sigma} c_{1/\beta} c_\beta(E)\, q^{2\alpha+\sigma}\, \|T\|_1\, C (\log_2 n)^{\beta-\alpha} \\ &\quad \times \left(\sum_{i=1}^{r} q^{-i(\alpha-\beta)s} (f(q^{i-1} \log_2 n))^{-s}\right)^{1/s}\end{aligned}$$

$$\leqslant 2^{3+\sigma} c_{1/\beta} c_\beta(E)\, q^{2\alpha}\, \|T\|_1\, C(\log_2 n)^{\beta-\alpha}$$

$$\times \left(\sum_{i=1}^{r} (q^{-i(\alpha-\beta-\sigma)\,s} (f(\log_2 n))^{-s} \right)^{1/s}$$

$$\leqslant c^{(3)}_{\alpha,\beta,f}(E)\, \|T\|_1\, C(\log_2 n)^{\beta-\alpha}\, (f(\log_2 n))^{-1},$$

where $c^{(3)}_{\alpha,\beta,f}(E) := 2^{3+\sigma} c_{1/\beta} c_\beta(E)\, q^{2\alpha} (\sum_{i=1}^{\infty} q^{-i(\alpha-\beta-\sigma)\,s})^{1/s}$. Hence we have proven

$$n^\beta (\log_2 n)^{\alpha-\beta} f(\log_2 n)\; e_{2n}(T) \leqslant c^{(4)}_{\alpha,\beta,f}(E)\, \|T\|_1 \sup_{k \leqslant n^\gamma \log_2(n+1)} k^\alpha f(k)\, e_k(K)$$

for $n \geqslant a$ and suitable $c^{(4)}_{\alpha,\beta,f}(E) \geqslant 1$. Now the assertion follows easily. ∎

Before we prove Theorem 3 we remark that

$$\sum_{i=1}^{n} i^{-b} e^{a \cdot i} \leqslant \frac{e^a}{a-b} (n+1)^{-b}\, e^{a \cdot n} \tag{8}$$

holds for all $0 < b < a$ and $n \in \mathbb{N}$.

Proof of Theorem 3. Again it suffices to consider the case $\varepsilon_1(K) \geqslant 1$. For fixed $\alpha > 0$ and $\gamma \geqslant 0$ we choose an integer a with $a \geqslant 2 + \max\{2^{4/\beta}, 2^{2/\alpha}\}$. Then for a fixed integer $n \geqslant a$ we define:

$$C := \sup_{j \leqslant n^{1+\beta/\alpha}} j^\alpha (\log_2(j+1))^\gamma\, \varepsilon_j(K)$$

$$r := \lfloor (\alpha+\beta) \log_2(n-1) \rfloor - 3$$

$$L := \lfloor \alpha \log_2(n-1) \rfloor$$

$$\alpha_i := \lfloor 2^{(i+3)/\alpha} + 1 \rfloor \qquad \text{for} \quad i = 0, ..., r$$

$$\beta_i := \max\{1, \alpha^\gamma\}\; C 2^{-(i+3)} (i+3)^{-\gamma} \qquad \text{for} \quad i = -2, ..., r.$$

Clearly, $a \leqslant n$ implies $1 \leqslant L \leqslant r$. Furthermore, since $\alpha_i \leqslant n^{1+\beta/\alpha}$ we obtain $\varepsilon_{\alpha_i}(K) < \beta_i$ for $0 \leqslant i \leqslant r$. Hence we can decompose T by Lemma 5 in

$$T = \sum_{i=0}^{L-1} T_i + \sum_{i=L}^{r} T_i + S$$

and receive

$$e_n(T) \leqslant e_{\lfloor n/2 \rfloor} \left(\sum_{i=0}^{L-1} T_i \right) + e_{\lfloor n/2 \rfloor} \left(\sum_{i=L}^{r} T_i \right) + \|S\|. \tag{9}$$

Before estimating the terms of the right side of (9), we observe that $4\beta_{-2} = 4\max\{1, \alpha^{\gamma}\}\, C2^{-1} \geqslant 2$ and thus $\|T_o\| \leqslant \|T\|_1 \leqslant 4\beta_{-2}\, \|T\|_1$. Hence for $\sigma > \beta$ and $0 \leqslant i \leqslant r$ we obtain

$$\begin{aligned}\|T_i\|\, \alpha_i^{\sigma-\beta} &\leqslant 4\beta_{i-2}\, \|T\|_1 \lfloor 2^{(i+3)/\alpha} + 1 \rfloor^{\sigma-\beta} \\ &\leqslant c_{\alpha,\beta,\gamma,\sigma}\, \|T\|_1\, C(i+1)^{-\gamma}\, 2^{(i+1)((\sigma-\beta)/\alpha-1)},\end{aligned} \tag{10}$$

where $c_{\alpha,\beta,\gamma,\sigma} := 2^{2+\sigma-\beta+2((\sigma-\beta)/\alpha)} \max\{1, \alpha^{\gamma}\}$.

To estimate the first term of inequality (9) we choose $\sigma > \alpha(\frac{\gamma}{\ln 2} + 1) + \beta$ and set $s := \frac{1}{1+\sigma}$. Then by Lemma 1 and inequality (10) we obtain

$$\begin{aligned}\left\lfloor \frac{n}{2} \right\rfloor^{\sigma} & e_{\lfloor n/2 \rfloor}\left(\sum_{i=0}^{L-1} T_i\right) \\ &\leqslant c_{1/\sigma}\left(\sum_{i=0}^{L-1} (\lambda_{1/\sigma,\infty}(T_i))^s\right)^{1/s} \\ &\leqslant c_{1/\sigma} c_{\beta,\sigma}(E)\left(\sum_{i=0}^{L-1} (\|T_i\|\, \alpha_i^{\sigma-\beta})^s\right)^{1/s} \\ &\leqslant c_{1/\sigma} c_{\alpha,\beta,\gamma,\sigma}(E)\, C\, \|T\|_1 \left(\sum_{i=1}^{L} i^{-\gamma s} 2^{i((\sigma-\beta)/\alpha-1)s}\right)^{1/s}\end{aligned} \tag{11}$$

for suitable $c_{1/\sigma} \geqslant 1$ and $c_{\alpha,\beta,\gamma,\sigma}(E) := c_{1/\sigma} c_{\beta,\sigma}(E)\, c_{\alpha,\beta,\gamma,\sigma}$. By inequality (8) there exists a constant $c^{(1)}_{\alpha,\beta,\gamma} > 0$ such that we may conclude

$$\begin{aligned}e_{\lfloor n/2 \rfloor}\left(\sum_{i=0}^{L-1} T_i\right) &\leqslant 3^{\sigma} c_{\alpha,\beta,\gamma,\sigma}(E)\, C\, \|T\|_1\, n^{-\sigma}\left(\sum_{i=1}^{L} i^{-\gamma s} 2^{i(\sigma-\beta)/\alpha-1)s}\right)^{1/s} \\ &\leqslant 3^{\sigma} c_{\alpha,\beta,\gamma,\sigma}(E)\, c^{(1)}_{\alpha,\beta,\gamma}\, C\, \|T\|_1 \\ &\qquad \times n^{-\sigma}(L+1)^{-\gamma}\, 2^{L((\sigma-\beta)/\alpha-1)} \\ &\leqslant c^{(1)}_{\alpha,\beta,\gamma}(E)\, C\, \|T\|_1\, n^{-\sigma} \\ &\qquad \times (\alpha \log_2(n-1))^{-\gamma}\, 2^{\alpha \log_2(n-1)((\sigma-\beta)/\alpha-1)} \\ &\leqslant \alpha^{-\gamma} 4^{\gamma} c^{(1)}_{\alpha,\beta,\gamma}(E)\, C\, \|T\|_1\, (\log_2(n+1))^{-\gamma}\, n^{-\alpha-\beta},\end{aligned}$$

where $c^{(1)}_{\alpha,\beta,\gamma}(E) := 3^{\sigma} c_{\alpha,\beta,\gamma,\sigma}(E)\, c^{(1)}_{\alpha,\beta,\gamma}$.

Next we estimate the second term of inequality (9). Analogously to (11) we get

$$\begin{aligned}\left\lfloor \frac{n}{2} \right\rfloor^{\sigma} e_{\lfloor n/2 \rfloor}\left(\sum_{i=L}^{r} T_i\right) &\leqslant c_{\alpha,\beta,\gamma,\sigma}(E)\, C\, \|T\|_1 \\ &\qquad \times \left(\sum_{i=L}^{r} (i+1)^{-\gamma s}\, 2^{(i+1)((\sigma-\beta)/\alpha-1)s}\right)^{1/s}\end{aligned}$$

for $\sigma := \beta + \frac{\alpha}{2}$ and $s := \frac{1}{1+\sigma}$. Thus we obtain

$$e_{\lfloor n/2 \rfloor}\left(\sum_{i=L}^{r} T_i\right) \leqslant 3^{\sigma} c_{\alpha,\beta,\gamma,\sigma}(E)\, C\, \|T\|_1\, n^{-\sigma} \times \left(\sum_{i=L}^{r} (i+1)^{-\gamma s}\, 2^{(i+1)((\sigma-\beta)/\alpha - 1)\, s}\right)^{1/s}$$
$$\leqslant c^{(2)}_{\alpha,\beta,\gamma}(E)\, C\, \|T\|_1\, n^{-\sigma}(L+1)^{-\gamma}\, 2^{L((\sigma-\beta)/\alpha-1)}$$
$$\leqslant 2^{1+\alpha+2\gamma}\alpha^{-\gamma} c^{(2)}_{\alpha,\beta,\gamma}(E)\, C\, \|T\|_1\, (\log_2(n+1))^{-\gamma}\, n^{-\alpha-\beta}$$

with $c^{(2)}_{\alpha,\beta,\gamma}(E) := 3^{\sigma} c_{\alpha,\beta,\gamma,\sigma}(E)(1/(1-\sqrt{2^s}))$. Finally we consider the last term. By Lemma 5 we obtain

$$\|S\| \leqslant 2\beta_{r-1}\, \|T\|_1$$
$$\leqslant 2^{3+\gamma} \max\{1, \alpha^{\gamma}\}\, C\, \|T\|_1\, (r+4)^{-\gamma}\, 2^{-(r+4)}$$
$$\leqslant c^{(2)}_{\alpha,\beta,\gamma}\, C\, \|T\|_1\, (\log_2(n+1))^{-\gamma}\, n^{-\alpha-\beta},$$

where $c^{(2)}_{\alpha,\beta,\gamma} := 2^{3+\alpha+\beta+3\gamma}(\alpha+\beta)^{-\gamma} \max\{1, \alpha^{\gamma}\}$. Combining the estimates we easily get the assertion. ∎

6. ENTROPY AND GELFAND WIDTHS OF CONVEX HULLS

In this section we give estimates for the nth entropy number $e_n(\mathrm{co}A)$ of a given precompact subset $A \subset E$ of a B-convex Banach space E with the help of Theorems 1, 2, and 3. The corollaries of this section complement earlier proven results of Carl *et al.* [6].

Moreover, since Theorems 1, 2, and 3 also hold for the Kolmogorov numbers, if E is a Hilbert space, we also obtain estimates for the Gelfand widths $c_n(\mathrm{co}A) := c_n(T_A)$, where $c_n(.)$ denotes the nth Gelfand number defined by

$$c_n(T\colon E \to F) := \inf\{\, \|T_{|E_o}\| : \operatorname{codim} E_o < n\}$$

and $T_A\colon \ell_1(A) \to E$ is the operator defined by $T_A(e_t) := t$ as pointed out in the introduction. For a geometric interpretation of $c_n(\mathrm{co}A)$ see, e.g., [12]. We start with a result which is analogous to Theorem 1.

THEOREM 4. *Let E be a Banach space, such that the dual space E' fulfills the condition* (3) *for the parameter $\beta \in (0, 1/2]$. Then for all $0 < \alpha < \beta$ there* *exists a constant $c_{\alpha,\beta}(E) \geqslant 1$, such that for all precompact $A \subset E$ and all $n \in \mathbb{N}$ the inequality*

$$\sup_{k \leqslant n} k^\alpha e_k(\mathrm{co} A) \leqslant c_{\alpha,\beta}(E) \frac{\|A\|}{\varepsilon_1(A)} \sup_{k \leqslant n} k^\alpha e_k(A)$$

holds. If E is a Hilbert space and $\beta = \frac{1}{2}$, this is also true for the Gelfand widths $c_n(\mathrm{co} A)$.

Proof. We consider the metric $d(x, y) := (1/\varepsilon_1(A)) \|x - y\|$ on A. One easily checks that $e_n((A, d)) = (1/\varepsilon_1(A))\, e_n(A)$ and $|T'_A: E' \to \ell_\infty((A, d))|_1 = \varepsilon_1(A)$. Hence we get $\varepsilon_1((A, d)) = 1$ and $\|T'_A: E' \to \ell_\infty((A, d))\|_1 = \|A\|$. Therefore we obtain

$$\begin{aligned} n^\alpha e_n(\mathrm{co} A) &\leqslant \sup_{k \leqslant n} k^\alpha e_k(T_A: \ell_1(A) \to E) \\ &\leqslant d_\alpha(E) \sup_{k \leqslant n} k^\alpha e_k(T'_A: E' \to \ell_\infty(A)) \\ &= d_\alpha(E) \sup_{k \leqslant n} k^\alpha e_k(T'_A: E' \to \ell_\infty((A, d))) \\ &\leqslant d_\alpha(E)\, c^{(1)}_{\alpha,\beta}(E) \frac{\|A\|}{\varepsilon_1(A)} \sup_{k \leqslant n} k^\alpha e_k(A) \end{aligned}$$

by Proposition 1 and Theorem 1. If E is a Hilbert space, the assertion for the Gelfand widths follows by $c_n(T_A) = d_n(T'_A)$ and the remarks of Section 3. ▮

Now we observe that in the situation of Theorem 4 we have

$$\sup_{k \leqslant n} k^\alpha e_k(A) \leqslant \sup_{k \leqslant n} k^\alpha e_k(\mathrm{co} A) \preccurlyeq \sup_{k \leqslant n} k^\alpha e_k(A)$$

and that in the Hilbert space case we have

$$\sup_{k \leqslant n} k^\alpha e_k(A) \leqslant \sup_{k \leqslant n} k^\alpha e_k(\mathrm{co} A) \preccurlyeq \sup_{k \leqslant n} k^\alpha c_k(\mathrm{co} A) \preccurlyeq \sup_{k \leqslant n} k^\alpha e_k(A)$$

by Carl's inequality. Hence, with the techniques used in the proof of Proposition 2 we obtain:

COROLLARY 3. *Let E be a Banach space of type $p > 1$ and $0 < \alpha < 1 - \frac{1}{p}$. If (a_n) is a positive sequence, such that $(n^\alpha a_n)$ is monotone increasing, then for every precompact subset $A \subset E$ we have*

$$e_n(A) \preccurlyeq a_n \quad \text{iff} \quad e_n(\mathrm{co} A) \preccurlyeq a_n$$

and

$$e_n(A) \sim a_n \qquad \textit{iff} \quad e_n(\mathrm{co}A) \sim a_n.$$

If E is a Hilbert space, these statements are also equivalent to $c_n(\mathrm{co}A) \preccurlyeq a_n$, *respectively* $c_n(\mathrm{co}A) \sim a_n$.

Remark. Corollary 3 states that in B-convex Banach spaces E the subsets A and $\mathrm{co}A$ paradoxically have the same entropy behaviour, whenever $(e_n(A))$ or $(e_n(\mathrm{co}A))$ decreases "slowly" in the above sense. This property of B-convex spaces is surprising and hard to understand since A can be very small, e.g., of type $A = \{x_n \mid n \in \mathbb{N}\}$, where $\|x_n\|$ "slowly" decreases. Moreover, as shown in [6], such a behaviour never occurs in $E = \ell_1$. Namely, there was shown that for the subset $A := \{a_n e_n \mid n \in \mathbb{N}\}$, where (e_n) denotes the canonical basis of ℓ_1 and (a_n) is an arbitrary positive, decreasing sequence such that $(n^\alpha a_n)$ is increasing for some $\alpha > 0$, we have

$$\varepsilon_n(A) \sim a_n \sim e_n(\mathrm{co}A).$$

In particular this shows that there is a wide difference by considering the entropy numbers of such subsets in ℓ_1 and in ℓ_p for $1 < p < \infty$.

THEOREM 5. *Let E be a Banach space, such that the dual space E′ fulfills the condition* (3) *for the parameter* $\beta \in (0, 1/2]$. *Furthermore let* $0 \leqslant \sigma < \alpha - \beta$ *and* $f\colon [0, \infty) \to (0, \infty)$ *be a function such that*

$$a^{-\sigma} f(x) \leqslant f(ax) \leqslant a^\sigma f(x)$$

holds for all $a, x \geqslant 1$. *Then there exists a constant* $c \geqslant 1$, *such that for all precompact* $A \subset E$ *and all* $n \in \mathbb{N}$ *the inequality*

$$\sup_{k \leqslant n} k^\beta (\log_2(k+1))^{\alpha-\beta} f(\log_2(k+1))\, e_k(\mathrm{co}A) \leqslant c\, \frac{\|A\|}{\varepsilon_1(A)} \sup_{k \leqslant a_n} k^\alpha f(k)\, e_k(A)$$

holds with $a_n := n^{\beta/(\alpha-\sigma)} \log_2(n+1)$. *If E is a Hilbert space and* $\beta = \frac{1}{2}$, *this is also true for the Gelfand widths* $c_n(\mathrm{co}A)$.

Proof. We define $C_n := \sup_{k \leqslant a_n} k^\alpha f(k)\, e_k(A)$. Then we obtain analogously to the previous theorem:

$$n^\beta (\log_2(n+1))^{\alpha-\beta} f(\log_2(n+1))\, e_n(T'_A\colon E' \to \ell_\infty(A)) \leqslant c^{(1)}_{\alpha,\beta,f}(E)\, \frac{\|A\|}{\varepsilon_1(A)}\, C_n.$$

Thus, for $\delta = 2\beta + \alpha + \sigma$ and $c_{\alpha,\beta,f}(E) := d_\delta(E)\, c^{(1)}_{\alpha,\beta,f}(E)$ we receive

$$\begin{aligned} n^\delta e_n(\mathrm{co}A) &\leqslant d_\delta(E) \sup_{k\leqslant n} k^\delta e_k(T'_A : E' \to \ell_\infty(A)) \\ &\leqslant d_\delta(E)\, c^{(1)}_{\alpha,\beta,f}(E) \frac{\|A\|}{\varepsilon_1(A)} \sup_{k\leqslant n} k^\delta k^{-\beta} \\ &\quad \times (\log_2(k+1))^{\beta-\alpha} f(\log_2(k+1))^{-1}\, C_k \\ &\leqslant c_{\alpha,\beta,f}(E) \frac{\|A\|}{\varepsilon_1(A)} C_n (\log_2(n+1))^\sigma f(\log_2(n+1))^{-1} \\ &\quad \times \sup_{k\leqslant n} k^{\delta-\beta} (\log_2(k+1))^{\beta-\alpha-\sigma} \\ &= c_{\alpha,\beta,f}(E) \frac{\|A\|}{\varepsilon_1(A)} C_n n^{\delta-\beta} (\log_2(n+1))^{\beta-\alpha} f(\log_2(n+1))^{-1}. \quad \blacksquare \end{aligned}$$

COROLLARY 4. *Let E be a Banach space of type $p>1$ and $\beta := 1 - \frac{1}{p}$. If $A \subset E$ is a precompact subset with $e_n(A) \preccurlyeq n^{-\alpha} (\log(n+1))^{-\gamma}$ for some $\alpha > \beta$ and $\gamma \in \mathbb{R}$, then we have*

$$e_n(\mathrm{co}A) \preccurlyeq n^{-\beta} (\log(n+1))^{-\alpha+\beta} (\log(\log(n+1)+1))^{-\gamma}.$$

This estimate is asymptotically optimal for $E = \ell_p$.

Proof. It remains to prove that the estimate is asymptotically optimal. Therefore we consider the set

$$A := \{(\log_2(n+1)^{-\alpha} (\log(\log(n+1)+1))^{-\gamma} e_n \mid n \in \mathbb{N}\} \subset \ell_p,$$

where $1 < p \leqslant 2$ and (e_n) is the canonical basis of ℓ_p. Then we obtain

$$e_n(A) \preccurlyeq n^{-\alpha} (\log(n+1))^{-\gamma}.$$

Now let $n \geqslant 2$ and $A_n := \{(\log_2(k+1)^{-\alpha} (\log(\log(k+1)+1))^{-\gamma} e_k \mid n \leqslant k \leqslant n^2\}$. Then by a result of Schütt in [14] we get

$$\begin{aligned} e_n(\mathrm{co}A) &\geqslant e_n(\mathrm{co}A_n) \\ &\geqslant (\log_2(n^2+1)^{-\alpha} (\log(\log(n^2+1)+1))^{-\gamma} e_n(id : \ell_1^{n^2-n} \to \ell_p^{n^2-n}) \\ &\succcurlyeq (\log_2(n+1))^{-\alpha} (\log(\log(n+1)+1))^{-\gamma} \left(\frac{\log_2\left(\frac{n^2-n}{n}+1\right)}{n} \right)^{1-1/p} \\ &\succcurlyeq n^{-\beta} (\log_2(n+1))^{\beta-\alpha} (\log(\log(n+1)+1))^{-\gamma}. \quad \blacksquare \end{aligned}$$

By the same methods Theorem 3 and Corollary 2 can be transferred to

THEOREM 6. *Let E be a Banach space, such that the dual space E' fulfills the condition* (3) *for the parameter $\beta \in (0, 1/2]$. Then for all $\alpha > 0$ and $\gamma \geqslant 0$ there exists a constant $c_{\alpha,\beta,\gamma}(E) \geqslant 1$, such that for all precompact $A \subset E$ and all $n \in \mathbb{N}$ the inequality*

$$\sup_{k \leqslant n} k^{\alpha+\beta}(\log_2(k+1))^{\gamma}\, e_k(\mathrm{co}A)$$
$$\leqslant c_{\alpha,\beta,\gamma}(E) \frac{\|A\|}{\varepsilon_1(A)} \sup_{k \leqslant n^{1+\beta/\alpha}} k^{\alpha}(\log_2(k+1))^{\gamma}\, \varepsilon_k(A)$$

holds. If E is a Hilbert space and $\beta = \frac{1}{2}$, this is also true for the Gelfand widths $c_n(\mathrm{co}A)$.

COROLLARY 5. *Let E be a Banach space of type $p > 1$, $\beta := 1 - \frac{1}{p}$. If $A \subset E$ is a precompact subset with $\varepsilon_n(A) \preccurlyeq n^{-\alpha}(\log(n+1))^{-\gamma}$ for some $\alpha > 0$ and $\gamma \in \mathbb{R}$, then we have*

$$e_n(\mathrm{co}A) \preccurlyeq n^{-\alpha-\beta}(\log(n+1))^{-\gamma}.$$

This estimate is asymptotically optimal for $E = \ell_p$.

Proof. Again, it suffices to show that the estimate is asymptotically optimal. Therefore we consider the set

$$A := \{n^{-\alpha}(\log_2(n+1))^{-\gamma}\, e_n \mid n \in \mathbb{N}\} \subset \ell_p$$

for $1 < p \leqslant 2$. Then we get $\varepsilon_n(A) = n^{-\alpha}(\log_2(n+1))^{-\gamma}$. Moreover, with $A_n := \{k^{-\alpha}(\log_2(k+1))^{-\gamma}\, e_k \mid k \leqslant n\}$ and the result of Schütt in [14] we obtain

$$\begin{aligned} e_n(\mathrm{co}A) &\geqslant e_n(\mathrm{co}A_n) \\ &\geqslant n^{-\alpha}(\log_2(n+1))^{-\gamma}\, e_n(id\colon \ell_1^n \to \ell_p^n) \\ &\geqslant n^{-\alpha}(\log_2(n+1))^{-\gamma}\, c_p \left(\frac{\log_2\left(\frac{n}{n}+1\right)}{n}\right)^{1-1/p} \\ &= c_p n^{-\alpha-\beta}(\log_2(n+1))^{-\gamma}. \quad \blacksquare \end{aligned}$$

Remark. Corollaries 4 and 5 also hold for the Gelfand widths $c_n(\mathrm{co}A)$ instead of the entropy numbers $e_n(\mathrm{co}A)$ if E is a Hilbert space. In this case it can be shown by a result in [10] that the estimates are asymptotically optimal.

With the help of Proposition 2 we finally obtain:

COROLLARY 6. *Let* $2 \leqslant p < \infty$ *and* $E = \ell_p$. *Then the estimates of Corollaries* 1 *and* 2 *are asymptotically optimal for suitable compact metric spaces* (K, d) *and suitable* 1-*Hölder-continuous operators* $T: E \to C(K)$.

Remark. For the limiting case $\alpha = \beta$ of Theorem 4 the estimate (5) can be transferred to

$$\sup_{k \leqslant n} (\log(k+1))^{-(1+\beta)} k^{\beta} e_k(\mathrm{co}A) \leqslant c_{\beta}(E) \frac{\|A\|}{\varepsilon_1(A)} \sup_{k \leqslant n} k^{\beta} e_k(A).$$

Hence we get $e_n(\mathrm{co}A) \preccurlyeq n^{-\beta}(\log(n+1))^{1+\beta}$ whenever $e_n(A) \preccurlyeq n^{-\beta}$. An estimate, where the log-term disappears, was shown in [12] for "small" sets A in Hilbert spaces. The general case is an open problem.

ACKNOWLEDGMENT

I thank my thesis advisor Professor B. Carl for his encouragement.

REFERENCES

1. J. Bourgain, A. Pajor, S. J. Szarek, and N. Tomczak-Jaegermann, On the duality Problem for entropy numbers of operators, *in* "Lecture Notes in Math.," Vol. 1376, pp. 50–63, Springer-Verlag, New York/Berlin, 1989.
2. B. Carl, Entropy numbers, *s*-numbers, and eigenvalue problems, *J. Funct. Anal.* **41** (1981), 290–306.
3. B. Carl, Inequalities of Bernstein–Jackson type and the degree of compactness of operators in Banach spaces, *Ann. Inst. Fourier* **35** (1985), 79–118.
4. B. Carl and D. E. Edmunds, Entropy of $C(X)$-valued operators and diverse applications, to appear.
5. B. Carl, S. Heinrich, and T. Kühn, *s*-numbers of integral operators with Hölder-continuous kernels over metric compacta, *J. Funct. Anal.* **81** (1988), 54–73.
6. B. Carl, I. Kyrezi, and A. Pajor, Metric entropy of convex hulls in Banach spaces, *J. London Math. Soc.*, to appear.
7. B. Carl and A. Pajor, Gelfand numbers of operators with values in a Hilbert space, *Invent. Math.* **94** (1988), 479–504.
8. B. Carl and I. Stephani, "Entropy, Compactness and the Approximation of Operators," Cambridge Univ. Press, London/New York, 1990.
9. M. Defant and M. Junge, Some estimates on entropy numbers, *Israel J. Math.* **84** (1993), 417–433.
10. A. Garnaev and E. Gluskin, On diameters of the Euclidean Sphere, *Dokl. Akad. Nauk SSSR* **277** (1984), 1048–1052.
11. U. Kotyczka, "Entropie-Eigenschaften kompakter metrischer Räume (X, d) und kompakter $C(X)$-wertiger Operatoren," Thesis, Jena, 1992.

12. W. V. Li and W. Linde, Metric entropy of convex hulls in hilbert spaces via majorizing measures, to appear.
13. G. Pisier, Remarques sur un résultat non publié de B. Maurey, *in* "Séminaire d'Analyse Fonctionnelle," École Polytechnique, Centre de Mathématiques, Palaiseau, 1980–1981.
14. C. Schütt, Entropy numbers of diagonal operators between symmetric Banach spaces, *J. Approx. Theory* **40** (1984), 121–128.